Algorithmics Research Paper Series

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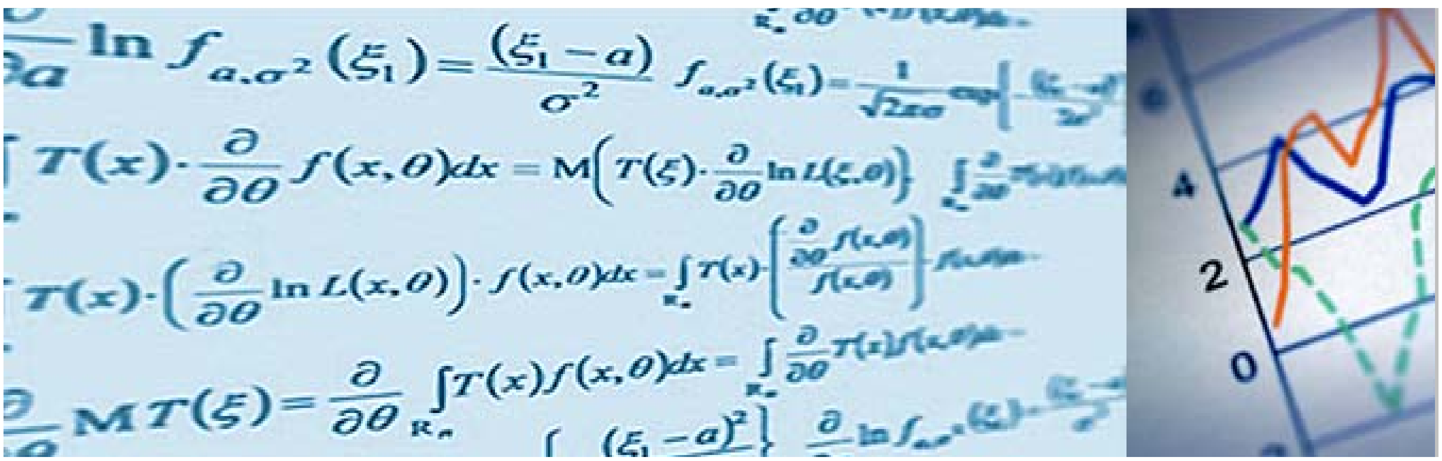
July 2007



Algorithmics

Valuation of CDO-Squared

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Abstract

In this paper, we consider the valuation of *CDO of CDOs or CDO–squared*, denoted CDO^2 in what follows. A traditional CDO contains a collateral pool of underlying credit risky securities, whose premium cash flows feed several tranches, each of which absorbs losses in accordance with its size and seniority. In CDO^2, the collateral pool contains tranches of CDOs. The complexity of the credit derivative contract (overlapping of the underlying names and the second tranche layer in the structure of the contract) does not seem to inspire an attempt of analytical valuation. Nevertheless, it appears to be possible to capture the correlation structure of the child tranches and to find approximation of the parent pool losses. Our method leads to an accurate estimation of the credit spread and has superior performance characteristics to that of Monte Carlo valuation of the CDO^2.

1 Introduction

In general, a CDO^2 is a credit derivative security whose underlying collateral is a pool of synthetic CDOs. The seller of the CDO^2 tranche sells partial protection to the buyer, by agreeing to absorb up to a set amount of the pool's losses, in exchange for periodic payments, often referred to as the tranche premiums.

In this spirit, the financial economics of a CDO^2 is similar to that of CDOs in that it can be described by the following standard pricing equation:

$$\mathbb{E}\left[\int_0^T e^{-\int_0^t r_s \, \mathrm{d}s} \, \mathrm{d}\mathbf{L}_t\right] = \mathbb{E}\left[\int_0^T e^{-\int_0^t r_s \, \mathrm{d}s} p_t \, \mathrm{d}t\right],$$

where $\mathbf{L}_t$ denotes the cumulative tranche loss by time t, p_t denotes the premium payment at that time, r_t is a short-rate process, $d(t) = e^{-\int_0^t r_s \, ds}$ is the discount factor and $\mathbb{E}$ denotes expectation with respect to a risk-neutral measure.

While similar in spirit to standard CDOs, the cumulative tranche losses suffered by a CDO^2 are more involved as they will be a function of complex dependencies between the underlying child pool losses (often referred to as 'name overlap') as well as underlying child tranche parameters (tranche attachment and detachment points, see [6]). These additional modeling complexities has resulted in elusive analytical expressions for CDO^2 valuation to date.

In this paper we consider a simple discrete-time model that allows us to find a closed-form solution to this problem. Our method is based on the factor model considered in [2], [7], [9].[1]

[1] A more general model using mixture of Gaussian copulas for pricing CDO^2 is considered in [10].

Based on the factor model above, current market practice for pricing and valuing a CDO^2 involves Monte Carlo simulation (MC) of name default events in the collateral pool. From the simulated default events in the pool, we can derive the child tranche losses and, ultimately, the parent tranche losses in order to value the transaction. While standard MC approaches are simple to understand and implement, they do come at the cost of poor computational performance. In fact, given that MC simulation occurs for each underlying name in the pool and default times are indexed for each payment date in the CDO^2, it is clear that the computational complexity of a CDO^2 transaction when using MC simulation is proportional to both the number of names in the pool, K, as well as payment dates, n, in the CDO^2:

$$\text{Complexity}(MC) = O(K \cdot n).$$

More recently, an alternative to pure MC simulation called Approximate Monte Carlo (AMC) has been proposed by Shelton in [12]. This method is based on the simulation of child tranche losses using a Normal approximation for the joint conditional distribution of the child pool losses. Based on the factor model above, the proposed algorithm works as follows: for each scenario on the credit driver and for each time step, compute the parameters of the joint distribution of child pool losses (based on a normal approximation) and generate a large sample of these vectors. From these vectors, compute the child tranche losses and, finally, the parent tranche losses.[2] In this case, the computational complexity of the AMC method can be represented as:

$$\text{Complexity}(AMC) = O(J \cdot n_X \cdot n)$$

where J is the number of child CDOs and n_X is the number of credit driver scenarios.

Since the number of child tranches is usually significantly smaller than the number of names in the pool, the proposed AMC method is often far more efficient computationally than MC simulation.

Other published analytical methods that can be used for CDO^2 pricing and valuation include the saddle-point method developed in [3] and a normal approximation method [5], [12]. This method was proposed for CDOs in [12] and for a special case of a CDO^2 absorbing losses of all child tranches in [5].

The purpose of this paper is to extend the normal approximation as represented in [5], [12] to include not only the estimation of the joint pool loss distribution but also the estimation of the joint child tranche loss distribution. Put simply, the idea behind this Normal Approximation of the Parent tranche loss method (NAP) is to use a conditional normal approximation for the joint child pool losses and to compute analytically the conditional variance and the conditional second moments of the child tranche losses (also based on a conditional normal approximation). Based on the NAP approach, we derive an analytical expression for the child tranche losses and derive an approximate analytic formula for the parent tranche loss.

Based on various examples, the NAP approach proves to be quite accurate for CDO^{2}s having a moderate number (say 6–20) of underlying child tranches. Since the approach by-passes the need for MC simulation all together, it also has the advantage of running orders of magnitudes faster than the MC method and 2–4 times faster than the AMC method.

The paper is organized as follows. In Section 2, we describe the model for the pricing and valuation of CDO^2. In Section 3, we review the conditional independence framework used for the valuation of the CDO^2 along with traditional CDOs. In Section 4 we describe the main analytical results

[2] In accordance with Formulae (2.2)–(2.4).

obtained for the CDO^2 losses under the assumption that the joint conditional distribution of the child pool losses and child tranche losses can be approximated by a multivariate Normal distribution. This assumption can be easily justified if the number of names in the collateral pool is large and the number of child tranches moderately large. In Section 5 we compare the results of the computation of the parent tranche spread obtained using our analytical results in Section 4, with that obtained by using the AMC and the plain MC methods. The paper concludes, in Section 6, with remarks related to the performance and accuracy of our proposed method. The proofs of some of the analytical results are deferred to the Appendix.

Acknowledgements

I would like to thank my colleagues Ian Iscoe, Asif Lakhany and Ben De Prisco for useful discussions and assistance in the development of this paper. I am also very grateful to Tom Hurd, Matheus Grasselli and Nizar Touzi for interesting comments and suggestions.

2 The pricing equation

We consider a discrete-time model for the pricing of a CDO^2. Let $\mathbb{T} = \{t_0 = 0, t_1, \ldots, t_i, \ldots, t_n = T\}$ denote the set of times, with T denoting the maturity of the CDO^2. The pool of securities contains K instruments (whose individual maturities are greater than T). Denote by N_k the recovery-adjusted notional value of the kth instrument initially in the pool, $k = 1, 2, \ldots, K$. The CDO^2 consists of J child CDOs, (enumerated 1, 2, ..., J). The tranches of these CDOs are called *child tranches*.

The (single) tranche of the CDO^2 will be called *the parent tranche* in what follows. Denote the size of the jth tranche, in monetary units, by $S^{(j)} = D^{(j)} - A^{(j)}$, $A^{(j)}$ being the protection level of the tranche, usually called *the attachment point*, and $D^{(j)}$ being the upper level of the absorbed jth CDO's pool loss usually called *the detachment point*. We also denote by A and D the attachment and detachment points of the parent tranche. In Figure 1, the name of the child tranche is framed in the box.

Denote by $L_i^{(j)}$ the cumulative losses up to time t_i of the jth pool. Then the jth child tranche loss by the time t_i is

$$\hat{L}_i^{(j)} = \min\left(S^{(j)}, \left(L_i^{(j)} - A^{(j)}\right)^+\right), \quad j = 1, 2, \ldots, J. \tag{2.1}$$

Let $I_{\mathcal{E}}$ denote the indicator of the event $\mathcal{E}$. Then we have

$$L_i^{(j)} = \sum_{k=1}^{K} N_k c_{kj} I_{\{\tau^{(k)} \le t_i\}}, \quad j = 1, 2, \ldots, J, \tag{2.2}$$

where c_{kj} determines the contribution of the kth name into the jth pool loss. We assume that

$$\sum_{j=1}^{J} c_{kj} = 1, \quad k = 1, 2, \ldots, K, \quad \text{and } c_{kj} \ge 0,$$

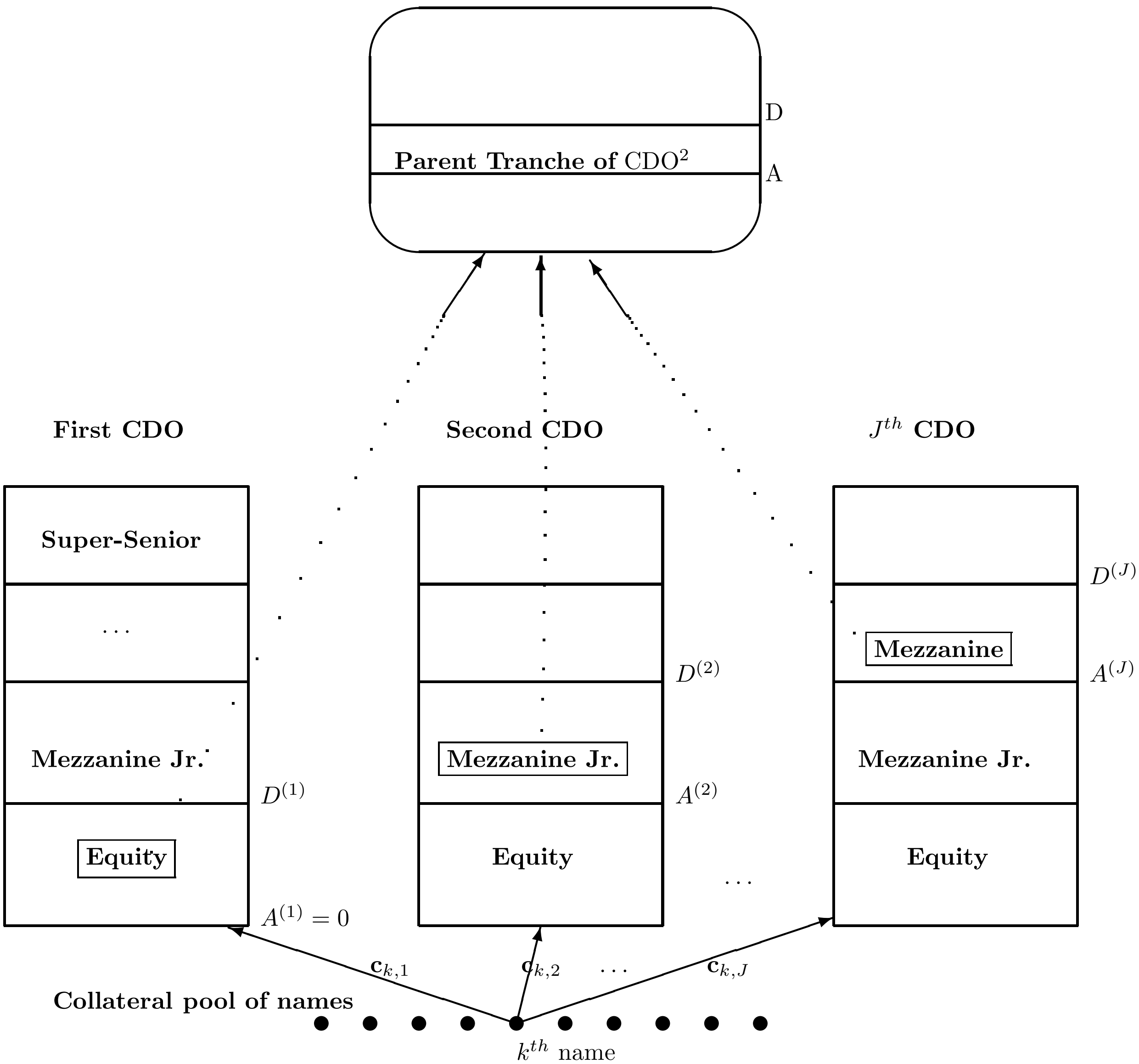

Figure 1: CDO^2 structure contains J child CDOs; framed tranches are sold to investors; the potential losses of the CDOs are partly absorbed by the synthetic parent tranche with the attachment point A and the detachment point D. The parameters $c_{k,1}, c_{k,2}, \ldots, c_{k,J}$ determine the contribution of the kth name into the loss of the child pools.

The CDO^2 inherits loss by the time t_i equal

$$\hat{\mathcal{L}}_i = \sum_{j=1}^{J} \hat{L}_i^{(j)} \tag{2.3}$$

from the child tranches. Then the parent tranche cumulative loss by the time t_i is

$$\mathcal{L}_i = \min\left(D - A, (\hat{\mathcal{L}}_i - A)^+\right). \tag{2.4}$$

Let d_i denote the discount factor corresponding to t_i, $i = 1, 2, \ldots, n$. The following equation (in which $\mathbb{E}$ denotes risk-neutral expectation) allows us to find the spread, s, of the parent tranche:

$$\mathbb{E}\left[\sum_{i=1}^{n}(\mathcal{L}_i - \mathcal{L}_{i-1})d_i\right] = \mathbb{E}\left[\sum_{i=1}^{n} s_i \cdot \Delta t_i \cdot (S - \mathcal{L}_i)\, d_i\right], \tag{2.5}$$

where $S = \sum_{j=1}^{J} S^{(j)}$. Computation of the right-hand side of Equation (2.5) is based on the following assumptions:

i. There is no replacement of the underlying instruments in the pool.
ii. The spread s_i of the CDO^2 tranche in the interval $(t_{i-1},\ t_i]$ may depend on time.
iii. The premium paid in the interval $(t_{i-1},\ t_i]$ by the CDO^2 parent tranche is proportional to the notional survived up to time t_i and the length of the interval, $\Delta t_i = t_i - t_{i-1}$. The notional is equal to $\sum_{j=1}^{J} S^{(j)} - \mathcal{L}_i$.
iv. The interest rate process is independent of the loss process.

Notice that the term $\mathbb{E}\left[s_i \cdot \Delta t_i \cdot (S - \mathcal{L}_i)\, d_i\right]$ represents the mean discounted premium paid by the CDO^2 at time t_i. Thus, the right-hand side of (2.5) represents the mean discounted premium paid during the lifetime of the CDO^2.

In the case where s_i does not depend on time, the spread is given by

$$s = \frac{\mathbb{E}\left[\sum_{i=1}^{n}(\mathcal{L}_i - \mathcal{L}_{i-1})d_i\right]}{\mathbb{E}\left[\sum_{i=1}^{n}(S - \mathcal{L}_i)\, d_i \cdot \Delta t_i\right]}. \tag{2.6}$$

In the case where the spreads are not to be calibrated, i.e., they are direct market inputs, the value of the CDO^2 at time t_i is given by the difference between the right- and left-hand sides of (2.5). The buyer's value of the CDO^2 is

$$V_{buy} = s \cdot \mathbb{E}\left[\sum_{i=1}^{n}(S - \mathcal{L}_i)\,\Delta t_i\, d_i\right] - \mathbb{E}\left[\sum_{i=1}^{n}(\mathcal{L}_i - \mathcal{L}_{i-1})d_i\right].$$

In both cases, the problem is reduced to computation of the mean absorbed parent tranche losses, $\mathbb{E}\mathcal{L}_i = \mathbb{E}\left[\sum_{j=1}^{J} L_i^{(j)}\right]$. To find a solution of the problem we briefly review, in the next section, the Gaussian copula model (see also [1], [2], [7], [8]) for the probabilities of default events.

3 Conditional independence framework

Let us consider a multi-factor Gaussian model with constant factor loadings for the default events of the names in the pool. Denote the risk-neutral, cumulative default probabilities of the kth name by $\hat{\pi}^{(k)}(t)$:

$$\mathbb{P}(\tau^{(k)} \le t) = \hat{\pi}^{(k)}(t), \qquad k = 1, 2, \ldots, K, \ t \in \mathbb{T}.$$

These probabilities can be determined from the CDS spread quotes.

Let X denote the vector of jointly normally distributed credit drivers $X := (X_1, \ldots, X_C)$ with each X_c standardized and denote by R the correlation matrix of X. Let $x = (x_1, \ldots, x_C)$ be a particular value of X. The conditional risk-neutral default probabilities are given by

$$\hat{\pi}^{(k)}(t, x) = \Phi\Big(\Big[\Phi^{-1}\left(\hat{\pi}^{(k)}(t)\right) - \sum_{c=1}^{C} \beta_c^{(k)} x_c\Big] / \sigma^{(k)}\Big), \tag{3.1}$$

where Φ denotes the standard normal cdf, and the factor-loadings $\beta_c^{(k)}$ and the idiosyncratic-risk volatilities, $\sigma^{(k)}$, satisfy the relation

$$(\sigma^{(k)})^2 + \sum_{c=1}^{C} \sum_{c'=1}^{C} \beta_c^{(k)} R_{cc'} \beta_{c'}^{(k)} = 1$$

for each $k = 1, 2, \ldots, K$.

In this framework, the default events of the names are conditionally independent. This assumption allows us to reduce the problem of computation of the expectations $\mathbb{E}\mathcal{L}_i$ to the case of independent names:

$$\mathbb{E}[\mathcal{L}_i] = \int_{\mathbb{R}^C} \mathbb{E}_x[\mathcal{L}_i] \, d\mu(x), \quad k = 1, 2, \ldots, K, \tag{3.2}$$

where $\mathbb{E}_x[\mathcal{L}_i]$ denotes the conditional expectation conditioned on $X = x$ (i.e., with default probabilities given by (3.1)) and μ is the joint distribution of the credit drivers. We denote by $\mathbb{P}_x$ the risk-neutral probability measure, conditional on $X = x$. Then, in a given scenario, x, $\mathbb{P}_x\left(\tau^{(k)} \le t_i\right) = \hat{\pi}_i^{(k)}(x)$.

4 Normal approximation method for CDO^2

Equations (2.3) and (2.6) show that the CDO^2 valuation problem can be decomposed into n separate problems of computation of the expected parent tranche losses. Our approach to that is based on a Normal approximation of the pool losses as well as the child tranche losses.

In [8], we proposed an idea of using a limit theorem describing convergence of the CDO loss to a compound Poisson process with time-dependent intensity. This approach to the valuation problem appeared to be efficient when the number of names in the CDO pool is about 50–300 while the risk-neutral default probabilities are less than 10%.

In a typical CDO^2 there are 10–20 child pools each of which contains about 100 names. Thus the total number of names in the CDO^2 underlying pool may be in the range 600 – 1500. That number of names is large enough to provide applicability of the Normal approximation of the underlying collateral pool losses.

The first version of the NAP method was developed for a special case of CDO^2 in [5]. In this section, we extend the NAP method to valuation of much more general CDO^2.

The main technical difficulty in the CDO^2 analysis is the overlapping effect: a name contributes simultaneously to losses of several child pools. In realistic examples of CDO^2, approximately 30–80% of the names contribute to all child pool losses in the CDO^2 transactions. Because of that child tranche losses become correlated even if the default events of the names are uncorrelated. For this reason, it is very important that analysis of the model captures the correlation structure of child's tranche losses for every time step.

The NAP method consists of two major steps common for all analytical methods in the conditional independence framework: the conditional analysis and unconditioning. The conditional analysis, in turn, is comprised of seven steps[3].

1. Computation of the jth child pool mean losses, $j = 1, 2, \ldots, J$.
2. Computation of the variance of the jth child pool losses.
3. Computation of the covariance matrix of the child pool losses.
4. Computation of the jth child tranche mean losses, $j = 1, 2, \ldots, J$ (see section 4.2).
5. Computation of the variance of the jth child tranche losses (see section 4.3).
6. Computation of the covariance matrix of the child tranche losses (see section 4.4).
7. Computation of the expected losses of the parent tranche (see section 4.5).

The first three steps provide parameters of the (conditional) Normal approximation of the child pool losses (see section 4.1). Steps 4–6 provide parameters of the child tranche losses. The final step is the computation of the expected parent tranche loss.

4.1 Approximation of the collateral pool losses

Let us fix both the index i of time step and the scenario on the credit driver, $X = x$. These parameters will not vary in this section. For this reason, we will simplify the notation for the conditional default probability, $\hat{\pi}_i^{(k)}(x)$, by ommiting the index i and the parameter x:

$$\pi_k = \hat{\pi}_i^{(k)}(x) \quad \text{as } i \text{ and } x \text{ are fixed.}$$

If the number of names in the underlying collateral pool, K, is large, the joint distribution of the vector of child's pool losses, $\hat{\mathbf{L}}_\mathbf{i} = (\hat{L}_i^{(1)}, \hat{L}_i^{(2)}, \ldots, \hat{L}_i^{(J)})$, can be approximated by a Normal distribution with the mean vector $\vec{a} = (a_1, \ldots, a_J)$ and the covariance matrix
$\text{cov}(\hat{\mathbf{L}}_\mathbf{i}) = \|\sigma_{jj'}\|_{j,j'=1}^J$, where

$$a_j = \sum_{k=1}^{K} c_{kj} N_k \pi_k, \quad j = 1, 2, \ldots, J \tag{4.1}$$

and the elements of the covariance matrix satisfy the relation

$$\sigma_{jj'} = \sum_{k=1}^{K} c_{kj} c_{kj'} \sigma_k^2, \quad j, j' = 1, 2, \ldots, J \tag{4.2}$$

in which the variance of the kth name cumulative losses by the time t_i is

$$\sigma_k^2 = {N_k}^2 \pi_k \cdot (1 - \pi_k), \quad k = 1, 2, \ldots, K. \tag{4.3}$$

[3] It is performed in each scenario on the latent variable and at each time step.

Equations (4.1)–(4.3) follow directly from Definition (2.2) of the cumulative child pool losses.

Remark. Convergence (under a natural normalisation condition) of the conditional distribution of the vector of cumulative losses, $\vec{L}_i = (L_i^{(1)}, L_i^{(2)}, \ldots, L_i^{(J)})$, to the multivariate Normal distribution with the mean vector $\vec{a}$ and the covariance matrix $\|\sigma_{jj'}\|_{j,j'=1}^{J}$ follows from the additive representation of the cumulative losses and from conditional independence of the default events.

Our next step is to describe the mean and the covariance of the child tranche losses.

4.2 Expected child tranche losses

The piecewise linear transformation of the jth child pool loss, $L_i^{(j)}$ into the tranche loss is described by Equation (2.1). Let us rewrite Equation (2.1) in the form

$$\hat{L}_i^{(j)} = (L_i^{(j)} - A^{(j)})^+ - (L_i^{(j)} - D^{(j)})^+, \tag{4.4}$$

where $A^{(j)}$ and $D^{(j)}$ are attachment and detachment points, respectively. Then we have

$$\mathbb{E}\hat{L}_i^{(j)} = \mathbb{E}(L_i^{(j)} - A^{(j)})^+ - \mathbb{E}(L_i^{(j)} - D^{(j)})^+. \tag{4.5}$$

Recall that if a random variable η has a Normal $\mathcal{N}(a, \sigma^2)$ distribution then the expected value $\psi(z, a, \sigma) = \mathbb{E}(\eta - z)^+$ is

$$\psi(z, a, \sigma) = (a - z) \cdot \overline{\Phi}\left(\frac{z-a}{\sigma}\right) + \sigma\, \varphi\left(\frac{z-a}{\sigma}\right), \tag{4.6}$$

where $\varphi(x)$ is the standard Normal probability density function, $\Phi(\cdot)$ is the standard Normal cumulative distribution function and $\overline{\Phi} = 1 - \Phi$. Since the distribution of $L_i^{(j)}$ is approximated by a Normal, we obtain from (4.5) and (4.6)

$$\mathbb{E}\hat{L}_i^{(j)} \approx \psi(A^{(j)}, a_j, \sigma(j)) - \psi(D^{(j)}, a_j, \sigma(j)), \tag{4.7}$$

where the mean of the jth pool loss is defined by (4.1) and the variance of the jth pool loss is

$$\sigma^2(j) = \sum_{k=1}^{K} c_{kj}^2 N_k^2 \pi_k (1 - \pi_k). \tag{4.8}$$

4.3 Variance of the child tranche loss

Our next step is to compute the second moment of the child tranche losses, $\mathbb{E}\left[\hat{L}_i^{(j)}\right]^2$.

Lemma 1 *Denote the centralized and normalized attachment and detachment points of the jth child tranche by*

$$\alpha_j = \frac{A^{(j)} - a_j}{\sigma(j)}, \quad \beta_j = \frac{D^{(j)} - a_j}{\sigma(j)}.$$

Then

$$\begin{aligned} \mathbb{E}\left[\hat{L}_i^{(j)}\right]^2 &= \sigma^2(j) \cdot \left((\beta_j - \alpha_j)^2 \overline{\Phi}(\beta_j) + (2\alpha_j - \beta_j)\varphi(\beta_j) - \alpha_j \varphi(\alpha_j)\right) \\ &+ \sigma^2(j)\,(1 + \alpha_j^2)(\Phi(\beta_j) - \Phi(\alpha_j)), \end{aligned} \tag{4.9}$$

where $\sigma(j)$ is defined by (4.8).

Lemma 1 is proved in the Appendix.

4.4 Covariance of the child tranche losses

Let us now compute the covariance matrix, $\|C_{jj'}\|$, $j, j' = 1, \ldots, J$, of the child tranche losses. Consider two tranches, j and j'. It is not difficult to see that the covariance of the child tranche losses $\hat{L}_i^{(j)}$ and $\hat{L}_i^{(j')}$ is

$$\begin{aligned} C_{jj'} &= \mathbb{E}\left(\left((L_i^{(j)} - A^{(j)})^+ - (L_i^{(j)} - D^{(j)})^+\right) \cdot \left((L_i^{(j')} - A^{(j')})^+ - (L_i^{(j')} - D^{(j')})^+\right)\right) \\ &- \mathbb{E}(L_i^{(j)} - A^{(j)})^+ \cdot \mathbb{E}(L_i^{(j')} - A^{(j')})^+ - \mathbb{E}(L_i^{(j)} - D^{(j)})^+ \cdot \mathbb{E}(L_i^{(j')} - D^{(j')})^+ \\ &+ \mathbb{E}(L_i^{(j)} - A^{(j)})^+ \cdot \mathbb{E}(L_i^{(j')} - D^{(j')})^+ + \mathbb{E}(L_i^{(j)} - D^{(j)})^+ \cdot \mathbb{E}(L_i^{(j')} - A^{(j')})^+. \end{aligned}$$

Thus, the computation of the covariance $\mathrm{cov}(\hat{L}_i^{(j)}, \hat{L}_i^{(j')})$ is reduced to computation of the expectations

$$\mathbb{E}\left((L_i^{(j)} - A^{(j)})^+ (L_i^{(j')} - A^{(j')})^+\right), \quad \mathbb{E}\left((L_i^{(j)} - A^{(j)})^+ (L_i^{(j')} - D^{(j')})^+\right),$$

and

$$\mathbb{E}\left((L_i^{(j')} - A^{(j')})^+ (L_i^{(j)} - D^{(j)})^+\right), \quad \mathbb{E}\left((L_i^{(j')} - D^{(j')})^+ (L_i^{(j)} - D^{(j)})^+\right).$$

Since all terms in the expectations have similar distributional structure, it is sufficient to consider the general expectation, $\mathbb{E}[(\zeta_1 - h_1)^+ \cdot (\zeta_2 - h_2)^+]$, where ζ_1 and ζ_2 are Normally distributed random variables. Denote their correlation coefficient by ρ, their variances by σ_i^2 and their expectations by a_i, $(i = 1, 2)$.

The random variables ζ_i can be represented in the form

$$\zeta_i = \sigma_i \xi_i + a_i, \quad i = 1, 2,$$

where ξ_i are the standard jointly Normal random variables with the covariance matrix

$$\Xi = \left\| \begin{array}{cc} 1 & \rho \\ \rho & 1 \end{array} \right\|. \tag{4.10}$$

We have

$$\mathbb{E}\big[(\zeta_1 - h_1)^+ \cdot (\zeta_2 - h_2)^+\big] = \sigma_1 \sigma_2 \cdot \mathbb{E}\big[(\xi_1 - z_1)^+ \cdot (\xi_2 - z_2)^+\big],$$

where

$$z_i = \frac{h_i - a_i}{\sigma_i}, \quad i = 1, 2.$$

The problem, therefore, is reduced to computation of the function

$$G(z_1, z_2, \rho) = \mathbb{E}\big[(\xi_1 - z_1)^+ \cdot (\xi_2 - z_2)^+\big], \tag{4.11}$$

where the joint distribution of the 2-dimensional vector (ξ_1, ξ_2) is the bivariate standard normal distribution function [4]

$$\Phi_\rho^{(2)}(x, y) = \frac{1}{2\pi\sqrt{1 - \rho^2}} \int_{-\infty}^{x} \int_{-\infty}^{y} \exp\left(-\frac{u^2 - 2\rho uv + v^2}{2(1 - \rho^2)}\right) \mathrm{d}u \, \mathrm{d}v.$$

[4]The marginal distribution of ξ_1 and ξ_2 is the standard normal $\mathcal{N}(0, 1)$.

Our next step is to show that the function $G(\cdot)$ can be represented as a linear combination of the functions φ, $\Phi(\cdot)$, $\Phi_\rho^{(2)}$ and some elementary functions of z_1 and z_2. We have

$$G(z_1, z_2, \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int\limits_{z_1}^{\infty}\int\limits_{z_2}^{\infty} (u - z_1)(v - z_2) \exp\left(-\frac{u^2 - 2\rho uv + v^2}{2(1-\rho^2)}\right) \mathrm{d}u\, \mathrm{d}v. \tag{4.12}$$

Proposition 1 *The function $G(z_1, z_2, \rho)$ in* (4.12) *satisfies the relation*

$$\begin{aligned} G(z_1, z_2, \rho) &= \sqrt{\frac{1-\rho^2}{2\pi}} \cdot \varphi(z_*) - z_1\varphi(z_2)\Phi\left(\frac{\rho z_2 - z_1}{\sqrt{1-\rho^2}}\right) - z_2\varphi(z_1)\Phi\left(\frac{\rho z_1 - z_2}{\sqrt{1-\rho^2}}\right) \\ &\quad + (z_1 z_2 + \rho)\left(1 - \Phi(z_1) - \Phi(z_2) + \Phi_\rho^{(2)}(z_1, z_2)\right), \end{aligned} \tag{4.13}$$

where

$$z_* = \sqrt{\frac{z_1^2 - 2\rho\, z_1 z_2 + z_2^2}{1-\rho^2}}.$$

Corollary 1 *The limit value of the function $G(z_1, z_2, \rho)$ as $\rho \to 1$ is*

$$G(z_1, z_2, 1) = (1 + z_1 z_2)\overline{\Phi}(z^*) - z_* \cdot \varphi(z^*),$$

where $z^* = max(z_1, z_2)$ and $z_* = \min(z_1, z_2)$. If $\rho \to -1$ and $z_1 + z_2 \le 0$ then

$$G(z_1, z_2, -1) = (z_1 z_2 - 1)\left(\Phi(-z_2) - \Phi(z_1)\right) - z_1\varphi(z_2) - z_2\varphi(z_1),$$

if $\rho \to -1$ and $z_1 + z_2 > 0$ then $G(z_1, z_2, -1) = 0$.

Proposition 1 and Corollary 1 are proved in the Appendix. Proposition 1 shows that an efficient computation of the conditional covariance of the child tranche losses requires a fast and accurate algorithm for computation of the standard bivariate Normal cumulative distribution function, $\Phi_\rho^{(2)}(\cdot,\ \cdot)$.

4.5 Parent tranche loss

Formula (4.13) allows us to compute the function $G(\cdot)$ and to find the covariance matrix $\|C_{jj'}\|$. Using Equation (2.4) we obtain the conditional mean, $a(x)$, and the conditional variance, $\sigma^2(x)$, of the (parent) CDO^2 loss, $\hat{\mathcal{L}}_i$, by the time t_i in the scenario $X = x$. We have

$$a(x) = \sum_{j=1}^{J} \mathbb{E}_x[\hat{L}_i^{(j)}]$$

and

$$\sigma^2(x) = \sum_{j=1}^{J} \sigma^2(\hat{L}_i^{(j)}) + 2\sum_{j<j'} C_{jj'}. \tag{4.14}$$

Approximating the conditional law of the parent CDO^2 loss by the Normal $\mathcal{N}(a(x), \sigma^2(x))$ distribution, we obtain the mean conditional loss of the parent tranche, $\mathbb{E}_x[\mathcal{L}_i]$:

$$\mathbb{E}_x[\mathcal{L}_i] \approx \psi(A, a(x), \sigma(x)) - \psi(D, a(x), \sigma(x)). \tag{4.15}$$

From (4.15) and (3.2) we obtain the approximation of the mean unconditional parent tranche loss

$$\mathbb{E}[\mathcal{L}_i] \approx \int_{\mathbb{R}^C} [\psi(A, a(x), \sigma(x)) - \psi(D, a(x), \sigma(x))] \, \mathrm{d}\mu(x), \tag{4.16}$$

where μ is the probability density function of the latent variable X.

Remark. The Normal Approximation method can be applied to the class of traditional synthetic CDOs. The accuracy of this method naturally depends on the number of names in the pool and the range of values of unconditional default probabilities (see also [12]).

5 Numerical examples

Since the method developed in this paper for CDO^2 is an approximation, a series of numerical experiments was performed for a number of CDO^2 structures to understand the limitations of the proposed method. In these experiments, we compared our analytical approximation with the results of Monte Carlo simulation of the CDO^2 and with that of the Approximate Monte Carlo method proposed in [12].

5.1 Parameters of the CDO^2 transaction

The CDO^2 transaction has 14 child CDOs. The collateral pool contains 1340 names. They are aggregated into the groups with common parameters in Table 1. Among the names in the pool 1200 of them have positive contribution vectors, therefore, simultaneously contributing to losses of all child tranches. The residual 140 names are partitioned into 14 groups each of which has 10 names contributing to a single child pool losses. The names in all groups have a common notional value, \$110M, and a common deterministic recovery rate, 40%. The second attribute is the number of names in the group. The 4th parameter in the table is β, the correlation with the single credit driver. This parameter takes values in the range [0.5, 0.7].

The risk-neutral default probabilities of the names are described in Table 2. The first column in this table is the index of the credit class; columns 2–6 represent the cumulative risk-neutral default probabilities of the names up to time $T_i = i$ years. The cumulative default probabilities are represented in the form

$$\pi^{(k)}(t) = 1 - \exp\left(-\Lambda_k(t)\right).$$

The values of the hazard rates, $\Lambda_k(T_i)$, are determined by the probabilities $\pi^{(k)}(T_i)$, $(k = 1, \ldots, K)$, $(i = 1, 2, \ldots, 5)$. The intermediate values of the cumulative default probabilities are obtained by linear interpolation of the hazard rates, $\Lambda_k(t)$. The structure of the child tranches is shown in Table 3.

The attachment points of the tranches vary in the range [0.06, 0.083] of the tranche notional; their detachment points are in the range [0.096, 0.1106]. The tranche thickness is in the range [2.3%, 3.97%] of the pool notional. These numbers are quite typical among CDO^2 transactions. The maturity of the contract, studied in our numerical experiments, $T = 5$ years. The deterministic interest rates

ν	$\mathbf{c}_{kj}$	β
800	$0.07, 0.07, \ldots, 0.07, 0.09$	0.5
400	$0.07, 0.07, \ldots, 0.07, 0.09$	0.62
10	1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0	0.7
10	0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0	0.7
10	0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0	0.7
10	0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0	0.7
10	0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0	0.7
10	0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0	0.7
10	0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0	0.7
10	0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0	0.7
10	0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0	0.7
10	0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0	0.7
10	0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0	0.7
10	0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0	0.7
10	0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0	0.7
10	0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1	0.7

Table 1: Parameters of the CDO^2 collateral pool: ν - number of names in the group; $N_k = 110$.

$\mathcal{C}$	T_1	T_2	T_3	T_4	T_5
1	0.0072	0.0185	0.0328	0.0495	0.0680
2	0.0074	0.0225	0.0439	0.0692	0.0967
3	0.026	0.0575	0.0931	0.1301	0.1607

Table 2: Risk-neutral cumulative default probabilities of the names in the CDO^2 collateral pool: $T_i = i$ (time is in years); $\mathcal{C}$ is a risk-neutral class of default probabilities.

are shown in Table 4. The number of premium dates $n = 20$, i.e. the premium is paid quarterly. The quarterly rates and the monthly rates are obtained by linear interpolation between the key nodes, $(T_i, r(T_i))$, $(i = 1, 2, \ldots, 5)$.

The numerical integration in the analytical valuation of the CDO^2 contract uses the quadrature proposed in [4], with $n_X = 20$ nodes. The same nodes are used as scenarios on the credit driver in the implementation of the AMC method.

5.2 Comparative analysis

In this section we compare the accuracy of the Normal approximation of the parent tranche spread with that obtained by the AMC and the plain MC methods. The CDO^2 transaction is described by the parameters represented in Tables 1–4. The parent tranche attachment point, and its detachment point, D, are measured as a percentage of the potential losses, $\mathcal{L}_P$, of the parent CDO. The potential parent CDO loss is the sum

$$\mathcal{L}_P = \sum_{j=1}^{J} \left(D^{(j)} - A^{(j)} \right).$$

j	$A^{(j)}$	$D^{(j)}$	j	$A^{(j)}$	$D^{(j)}$
1	7.6%	10%	8	6.3%	9.6%
2	6.7%	10.1%	9	6.3%	9.7%
3	6.6%	10.3%	10	6.03%	10.6%
4	7.5%	10.2%	11	6.03%	10.6%
5	6.3%	9.6%	12	7.3%	11.06%
6	6.13%	9.7%	13	8.3%	10.6%
7	6.3%	9.6%	14	6.0%	9.6%

Table 3: Attachment and detachment points of the child tranches: j is the index of the child tranche, $A^{(j)}$ is the attachment point of the jth child tranche, $D^{(j)}$ is its detachment point

Parameter	T_1	T_2	T_3	T_4	T_5
$r(T_i)$	0.046	0.05	0.056	0.058	0.06
d_i	0.9550	0.9048	0.8454	0.7929	0.7408

Table 4: Discount curve and discount factors.

The results of comparison are shown in Table 5. This table contains the non-parametric 95% confidence interval on the credit spread of the parent tranche of the CDO^2 obtained by the MC and AMC methods. The results (in basis points) are rounded to the nearest integer in the table. The standard MC method and the AMC method used 100 experiments to determine the 95% confidence interval. The number of random paths in each experiment is 10^4.

The first two columns represent the attachment and detachment points of the parent tranche; the third and the fifth columns represent the 95% nonparametric confidence interval computed using the AMC and the plain MC methods. The fourth column represents the results of application of the analytical method to the computation of the credit spread. In this experiment, we also compare

A	D	AMC	NAP	MC
0%	10%	$[61, 62]$	62	$[60, 65]$
5%	15%	$[57, 58]$	58	$[56, 60]$
10%	20%	$[55, 55]$	55	$[52, 58]$
20%	30%	$[51, 52]$	51	$[48, 55]$
30%	40%	$[48, 48]$	48	$[44, 52]$
40%	50%	$[45, 45]$	45	$[42, 49]$
50%	60%	$[42, 42]$	42	$[38, 47]$
60%	70%	$[40, 40]$	40.5	$[37, 45]$
70%	80%	$[39, 39]$	39	$[36, 42]$

Table 5: Credit spread (bp) of the parent tranche; A - attachment point of the tranche, D is its detachment point.

relative performance of the Monte Carlo methods with respect to the NAP method. The performance of a method, m, is measured as the ratio

$$P_m = \frac{T_*}{T_m}, \tag{5.1}$$

where T_* is the mean computation time of the NAP method and T_m is the mean computation time of the method m. In the first experiment, T_*, T_{AMC} and T_{MC} do not depend on the attachment and detachment points of the parent tranche. The performance of the AMC method is 26%; the performance of the MC method is only 0.5%.

The results in Table 5 demonstrate that the AMC and the NAP methods have superior accuracy with respect to the standard MC algorithm. The confidence intervals for the credit spread obtained by the plain MC method, have lengths as large as 8 bp while the maximal length of the confidence interval of the AMC method is only 1 bp after rounding. In terms of performance, the MC method is approximately 200 times slower than the NAP method; the AMC method is almost 4 times slower than the NAP method.

We describe now the results of the second numerical experiment in which we study the accuracy of approximation of the parent tranche spread in the case the spread of the parent tranche is high. In this experiment, the number of names in the collateral pool is $K = 1400$, the structure (list of names in the collateral pool, their contributions to the child pools and child tranche parameters) are shown in Tables 6-7. The new set of parameters provides the par spread significantly higher than in the first experiment.

$\mathbf{N}_k$	**k**	β_k
100	$1 - 200$	0.55
150	$201 - 300$	0.75
150	$301 - 400$	0.65
140	$401 - 500$	0.71
140	$501 - 600$	0.6
130	$601 - 700$	0.8
130	$701 - 800$	0.45
120	$801 - 900$	0.52
120	$901 - 1000$	0.74
125	$1001 - 1100$	0.65
125	$1101 - 1200$	0.47
110	$1201 - 1400$	0.68

Table 6: Combinations of notionals and correlations in CDO^2 collateral pool: k - index of a name in the pool, $(1 \le k \le 1400)$; notionals are in millions \$.

The recovery rate of all names in the pool is 40%. The contribution vectors are described in Table 7. There are 11 different contribution vectors, $c^{(g)} \in \mathbb{R}^{10}$, $g = 1, 2, \ldots, 11$. The set of indices, k, corresponding to the gth contribution vector is represented in the form[5]

$$k = 200 \cdot (r - 1) + s, \quad r = 1, 2, \ldots, 7; \quad g = 1, 2, \ldots, 11, \tag{5.2}$$

where

$$s = s_{gh} = \begin{cases} 10(g-1) + h, & \text{with } 1 \le g, h \le 10 \text{ for } 1 \le s \le 100, \\ 100 + h, & \text{with } g = 11, 1 \le h \le 100 \text{ for } 101 \le s \le 200. \end{cases}$$

If k satisfies (5.2) with $s = s_{gh}$ then the vector of contributions is the 10-vector in the gth row of Table 7. In this experiment, the CDO^2 has $J = 10$ child tranches. Their attachment and detachment

[5] Each k, $1 \le k \le 1400$ can be uniquely represent in the form (5.2).

g	$\mathbf{c}^{(g)}$	$\mathbf{s}_g$
1	$(0.5, 0.5, 0, 0, \ldots, 0)$	$s_g = 1, 2, \ldots, 10$
2	$(0, 0.5, 0.5, 0, \ldots, 0)$	$s_g = 11, 12, \ldots, 20$
3	$(0, 0, 0.5, 0.5, 0, \ldots, 0)$	$s_g = 21, 22, \ldots, 30$
...	...	...
9	$(0, 0, \ldots, 0.5, 0.5)$	$s_g = 81, 82, \ldots, 90$
10	$(0.5, 0, \ldots 0, 0.5)$	$s_g = 91, 92, \ldots, 100$
11	$(0.1, 0.1, \ldots, 0.1)$	$s_g = 101, 102, \ldots, 200$

Table 7: Vector of contributions of the names in CDO^2 collateral pool.

points are shown in Table 8. This table also contains the expected child pool losses at the maturity of the contract. The attachment and detachment points of the parent tranche are $A = 150M$ and

j	$A^{(j)}$	$D^{(j)}$	$\mathbb{E}[L_n^{(j)}]$	j	$A^{(j)}$	$D^{(j)}$	$\mathbb{E}[L_n^{(j)}]$
1	196.0	376.0	219.5	6	187.5	247.5	241.63
2	187.5	247.5	236.25	7	158.7	240.0	240.65
3	187.5	247.5	295.65	8	190.0	370.0	254.31
4	174.5	234.5	323.76	9	187.5	247.5	253.50
5	187.5	247.5	278.79	10	180.5	247.5	234.13

Table 8: Attachment and detachment points of child tranches and expected child pool losses at maturity (Millions $).

$D = 300M$. The discount curve has 20 nodes corresponding to the payment dates of the contract. The risk-neutral default probabilities are described in Table 2.

The first measurement is related to the accuracy and the relative performance of the methods. The results for the par spread of the CDO^2 are summarized in Table 9. The number of paths in the MC method and in the AMC method is $15,000$. The 95% nonparametric confidence interval is computed from the results of 100 experiments.

$Method$	$\overline{s}$	CI	L_{ci}(bp)	P	ε
AMC	780.4	[779, 781]	2	24.9%	0.94%
NAP	787.5	–	–	100%	–
MC	780.5	[760, 800]	40	0.7%	0.93%

Table 9: Par spread of the CDO^2: $\overline{s}$ - mean spread, CI - 95% confidence interval, L_{ci} - length of confidence interval, P - relative performance, and relative error (ε) of NAP method.

The results of the MC estimation of the par spread and the result obtained by the AMC method are very close in our experiments: the difference between the values of the spread is less than 1 bp. Since the MC method is not sensitive to the number of nodes in the quadrature used in both the NAP method and the AMC method, we can conclude that that number of nodes is sufficient for the CDO^2 analyzed in this section.

Let us now analyze how the error of approximation of the par spread depends on the attachment and detachment points of the parent tranche. In Table 10 we collect the results of several numerical experiments demonstrating that the relative error stays at the same level, below 1%.[6] The number of paths in the AMC method is, as before, $15,000$; the number of nodes in the quadrature is, again, $n_X = 20$.

A	D	AMC(bp)	NAP(bp)	$\varepsilon(\%)$
50	150	988.9	986.0	0.32
100	200	877.7	882.1	0.51
150	300	780.4	787.5	0.94
250	400	694.0	698.2	0.62
350	500	626.7	626.0	0.01
550	700	506.2	502.7	0.83
700	900	329.6	326.4	0.97

Table 10: Relative error of approximation of the par spread; attachment and detachment points are in millions $.

The relative error of the NAP method for approximation of the CDO^2 par spread is not, apparently, a monotone function of the attachment point, A, of the parent tranche. Additional experiments would be required to understand this functional dependency.

Now we are in position to analyze the dependency of the relative error, ε, on the number of child tranches, J. We consider the following CDO^2 structure: the number of names is $K = 1400$, the child tranches have the same attachment and detachment points, $A^{(j)} = 187.5M$, $D^{(j)} = 247M$. The vector of contributions, c_{kj}, $(j = 1, 2, \ldots, J)$ has coordinates $c_{kj} = J^{-1}$, $(j = 1, 2, \ldots, J)$, $(k = 1, 2, \ldots, K)$. The number of child tranches is in the range $1 \le J \le 16$.

In all numerical experiments the difference (in par spread) between the MC and AMC methods was negligible. For this reason, we focus on comparison of accuracy and performance of the AMC and NAP methods for estimation of the par spread.

Figure 2 represents relative error of the NAP method, ε, and the performance of the AMC method, P, measured as a percentage of the performance of the NAP method (see Equation (5.1)). The relative error of the analytical approximation decreases as the number of child tranches, J, increases. At the same time, the performance of the NAP method degrades as J increases.

This behaviour of the relative performance of the NAP method is explained by the quadratic term, J^2, in the formula for the complexity of the NAP method and linear dependency on J of the complexity of the AMC method:

$$\text{Complexity}(NAP) = O\left(J^2 \cdot n \cdot n_X\right).$$

The quadratic term enters because at every time step and in each scenario we compute the covariance matrix of the child tranche losses.

If the number of child tranches, $J > 20$, the AMC method becomes more efficient than the NAP method.

[6]The performance of the AMC and the MC methods is exactly the same as in Table 9 in our experiments.

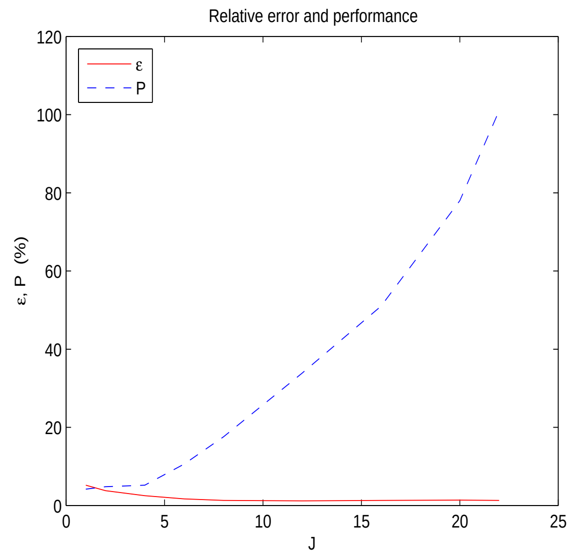

Figure 2: Comparison of NAP and AMC methods.

The relative error of the NAP method decreases as a function of J since the more child tranches are in the CDO^2, the better the conditional Normal approximation captures the expected losses of the parent tranche. In our experiments, the relative error decreases from 5.4% to 1.3% as the number of child tranches changes from 1 to 22.

6 Conclusion

The Normal Approximation method, proposed in this paper for the analytical valuation of a CDO^2 without cross-subordination, represents an alternative to both the AMC and the plain MC methods. The NAP method acounts for the dependency between the default events and overlapping of the loss contributions of the names in the collateral pool. The method can be generalized for the CDO^2 with cross-subordination.

Both the NAP method and the AMC method are much more efficient than the MC method. The relative performance of the NAP and the AMC methods depends on the number of time steps in the computation of the contract value. The relative error of the NAP method depends on the number of child tranches. For this reason, it is preferable to implement a flexible computational strategy combining both the AMC and the NAP methods, with selection on an instrument-by-instrument basis.

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A Proof of the results

A.1 Proof of Lemma 1

The losses of the jth child tranche are

$$\hat{L}_i^{(j)} = \min\left(D^{(j)} - A^{(j)},\ \max(0, L_i^{(j)} - A^{(j)})\right).$$

The distribution of the random variable $L_i^{(j)}$ is approximated by the Normal distribution $\mathcal{N}(a_j, \sigma_j^2)$. The cdf of $\mathcal{N}(a, \sigma^2)$ is denoted by $\Phi_{a,\sigma}(\cdot)$. Obviously,

$$\Phi_{a,\sigma}(x) = \Phi\left(\frac{x-a}{\sigma}\right).$$

Then we have

$$\mathbb{E}\left[\hat{L}_i^{(j)}\right]^2 = \int_{A^{(j)}}^{D^{(j)}} (x - A^{(j)})^2 \, \mathrm{d}\Phi_{a_j,\sigma_j}(x) + \int_{D^{(j)}}^{\infty} (D^{(j)} - A^{(j)})^2 \, \mathrm{d}\Phi_{a_j,\sigma_j}(x),$$

and using the identities

$$\begin{aligned}
\int_{\alpha}^{\beta} z \, \mathrm{d}\Phi(z) &= \varphi(\alpha) - \varphi(\beta), \\
\int_{\alpha}^{\beta} z^2 \, \mathrm{d}\Phi(z) &= \Phi(\beta) - \Phi(\alpha) - (\beta\varphi(\beta) - \alpha\varphi(\alpha)),
\end{aligned}$$

we obtain Equation (4.9). ■

A.2 Proof of Proposition 1

From (4.12) we have

$$G(z_1, z_2, \rho) = I_{11}(z_1, z_2, \rho) - z_1 I_{01}(z_1, z_2, \rho) - z_2 I_{10}(z_1, z_2, \rho) + z_1\, z_2\, I_{00}(z_1, z_2, \rho), \tag{A.1}$$

where

$$I_{11}(z_1, z_2, \rho) = \int_{z_1}^{\infty} \int_{z_2}^{\infty} xy \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2 - 2\rho\, xy + y^2}{2(1-\rho^2)}\right) \mathrm{d}x\, \mathrm{d}y, \tag{A.2}$$

$$I_{10}(z_1, z_2, \rho) = \int_{z_1}^{\infty} \int_{z_2}^{\infty} x \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2 - 2\rho\, xy + y^2}{2(1-\rho^2)}\right) \mathrm{d}x\, \mathrm{d}y, \tag{A.3}$$

$$I_{01}(z_1, z_2, \rho) = \int_{z_1}^{\infty} \int_{z_2}^{\infty} y \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2 - 2\rho\, xy + y^2}{2(1-\rho^2)}\right) \mathrm{d}x\, \mathrm{d}y \tag{A.4}$$

and

$$I_{00}(z_1, z_2, \rho) = \int_{z_1}^{\infty} \int_{z_2}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2 - 2\rho\, xy + y^2}{2(1-\rho^2)}\right) \mathrm{d}x\, \mathrm{d}y, \tag{A.5}$$

The integrals in (A.2)–(A.5) represent moments of the truncated bivariate Normal distribution. Their values can be obtained using integration by parts (see [11]):

$$
\begin{aligned}
I_{11}(z_1, z_2, \rho) &= \rho \cdot \left(z_1\,\varphi(z_1)\Phi\left(\frac{\rho z_1 - z_2}{\sqrt{1-\rho^2}}\right) + z_2\,\varphi(z_2)\Phi\left(\frac{\rho z_2 - z_1}{\sqrt{1-\rho^2}}\right)\right) \\
&\quad + \rho \cdot \Big(1 - \Phi(z_1) - \Phi(z_2) + \Phi^{(2)}_{\rho}(z_1, z_2)\Big) + \sqrt{\frac{1-\rho^2}{2\pi}} \cdot \varphi(z_*), \qquad \text{(A.6)} \\
I_{10}(z_1, z_2, \rho) &= \rho \cdot \varphi(z_2)\,\Phi\left(\frac{\rho z_2 - z_1}{\sqrt{1-\rho^2}}\right) + \varphi(z_1)\,\Phi\left(\frac{\rho z_1 - z_2}{\sqrt{1-\rho^2}}\right), \qquad \text{(A.7)} \\
I_{01}(z_1, z_2, \rho) &= \rho \cdot \varphi(z_1)\,\Phi\left(\frac{\rho z_1 - z_2}{\sqrt{1-\rho^2}}\right) + \varphi(z_2)\,\Phi\left(\frac{\rho z_2 - z_1}{\sqrt{1-\rho^2}}\right), \qquad \text{(A.8)} \\
I_{00}(z_1, z_2, \rho) &= 1 - \Phi(z_1) - \Phi(z_2) + \Phi^{(2)}_{\rho}(z_1, z_2). \qquad \text{(A.9)}
\end{aligned}
$$

From formulae (A.6)–(A.9) and Equation (A.1) we derive (4.13). ■

A.3 Proof of Corollary 1

Let us prove the first statement of Corollary 1. If $\rho = 1$, $\xi_1 = \xi_2$. Then we have

$$
\begin{aligned}
\mathbb{E}\big[(\xi_1 - z_1)^+ \cdot (\xi_2 - z_2)^+\big] &= \int_{z^*}^{\infty} (x - z_1)(x - z_2)\varphi(x)\,\mathrm{d}x \\
&= 1 - \Phi(z^*) + z^*\varphi(z^*) - z_1\varphi(z^*) - z_2\varphi(z^*) + z_1 z_2 \overline{\Phi}(z^*) \\
&= (1 + z_1 z_2)\overline{\Phi}(z^*) + (z^* - z_1 - z_2)\varphi(z^*).
\end{aligned}
$$

Since $z^* - z_1 - z_2 = -\min(z_1, z_2)$ we obtain

$$
G(z_1, z_2, 1) = (1 + z_1 z_2)\overline{\Phi}(z^*) - \min(z_1, z_2)\varphi(z^*).
$$

The continuity of the function $G(z_1, z_2, \rho)$ at $\rho = 1$ follows from smoothness of the coefficients of the Cholesky decomposition of the matrix Ξ. ■

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