Understanding Stochastic Exposures and LGDs in Portfolio Credit Risk

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This paper presents a case study on the impact of stochastic exposures and losses given default (LGD) on portfolio credit-risk estimation. In this sense, four factors have a substantial effect on credit losses: exposure (market) volatility, credit correlations, market-credit correlations, and portfolio granularity. We emphasize the importance of treating stochastic exposures for economic and regulatory capital properly. In particular, we discuss the limitations of the regulatory proposals when market correlations affect exposures/LGDs and when market and credit risk are correlated. Correlated exposures/LGDs and market-credit correlations occur quite frequently and are of sizeable proportions; the latter are the cause of wrong-way exposures. Although the examples in this paper use portfolios of derivatives, the techniques and results apply equally to other cases where LGDs, exposures and spreads are stochastic.

As the culminating component of an enterprise credit-risk management framework, portfolio credit-risk analysis has been the focus of intensive research. Over the last decade, the industry developed the first generation of portfolio credit-risk models, including CreditMetrics, CreditRisk+, Credit Portfolio View and KMV's Portfolio Manager (see Gupton et al. 1997, Credit Suisse Financial Products 1997, Wilson 1997a and 1997b, Kealhofer 1996). Various researchers have described a general framework for portfolio credit-risk modelling that highlights an underlying mathematical equivalence among these models (see, for example, Koyluoglu and Hickman 1998, and Gordy 2000). This common framework is at the centre of an internal ratings-based approach in the recommendations of the Basel Committee (c.f. Basel Committee on Banking Supervision 2001, Gordy 2001, Wilde 2001).

A major limitation of first-generation portfolio models is the treatment of credit risk isolated from other market factors, such as interest rates, foreign exchange rates and spreads. By making the implicit assumption that these market risk factors are deterministic, such models cannot account properly for stochastic, and perhaps correlated, exposures. They also do not incorporate correlations between collateral and loss given default (LGD) with credit events.

These assumptions can have a great impact on the capital of portfolios that contain derivatives (such as swaps and options), on portfolios with collateral or other credit-mitigation techniques, and even on loans. Furthermore, these models cannot estimate losses that arise jointly from both discrete credit events and market factors (e.g., spread) changes. Such limitations can only be overcome by using an integrated market- and credit-risk framework. In response to these limitations, practitioners have started to develop and extend portfolio models that explicitly address stochastic (and correlated) exposures and LGDs in portfolio models (Iscoe et al. 1999, Frye 2000a and Frye 2000b, Bürgisser et al. 2001).

Practitioners have long recognized the need for integrating market and credit risk in the trading book, where counterparty credit exposures of derivatives portfolios have a strong dependence on the level of the various market factors. Many financial institutions are thus implementing counterparty exposure measurement and limits systems based on advanced simulation methods. Simulation methods can capture the stochastic nature of exposures accurately, as well as the richness of netting agreements, collateral behaviour and mitigation techniques that are commonly used (see, e.g., Aziz and Charupat 1998, Aguais and Rosen 2001). Simulation models can also be used effectively to capture wrong-way exposures, roll-off risk and settlement risk. Iscoe et al. (1999) further explain that it is natural to integrate multi-step counterparty exposure simulations into the general, portfolio credit-risk framework using the basic concepts of Mark-to-Future (Aguais et al. 2000, Dembo et al. 2000, Aguais and Rosen 2001).

While there is wide consensus on the applicability of advanced exposure methods to measure and control counterparty risk of derivatives portfolios, the industry and regulators are discussing *how* they may be applied for regulation. The current BIS regulatory model attempts to capture potential exposures of derivatives through simplistic add-on factors by multiplying the notional of each transaction (Basel Committee on Banking Supervision 1988). Although simple to implement, add-ons have been criticized widely for not capturing accurately the stochastic nature of these exposures in the future. Accordingly, many practitioners are recommending that regulators allow for internal models based on more sophisticated methods, such as simulation, to be used as the basis of regulatory capital calculations. For example, the recommendation put forth by the International Swaps and Derivatives Association (ISDA) 2001 is to allow for internal models to estimate the *loan-equivalent exposures* in capital calculations. More precisely, the paper recommends using the *expected counterparty exposure* over one year (netted and with the proper application of collateral and mitigation) as the measure of loan-equivalent exposure.

This paper illustrates, through a simple case study, the potential impact of stochastic exposures on portfolio credit-risk estimation, and its implications on economic and regulatory capital. Although we use portfolios of derivatives that naturally present stochastic exposures as examples, the techniques and results apply equally to other cases where LGDs, exposures and spreads are stochastic. We discuss some limitations of the regulatory proposals when exposures/LGDs are correlated (market correlations), and when market and credit risks are correlated. Exposure/LGD correlations can be significant, for example, in derivatives portfolios and collateralized portfolios; correlated LGDs may also arise naturally as a consequence of systemic economic conditions. Market and credit correlations may also arise frequently and are of sizeable proportions in the case of wrong-way exposures.

The rest of the paper is organized as follows: first, we introduce the simple portfolio model used in the analysis. The case study follows, introducing the portfolio and describing the counterparty exposure profiles. Then, we present the results for a finite portfolio. We present various examples that show the impact of stochastic exposures, their correlations and market-credit correlations. Thereafter, we assess the impact of stochastic exposures as portfolios become larger and converge to the systemic losses. The paper concludes with some final remarks.

Portfolio model

To asses the impact of stochastic exposures, and as is consistent with common practice, we consider a single-step portfolio credit-risk calculation over one year. We use the integrated market and credit-risk portfolio framework in Iscoe et al. (1999). For simplicity, we focus on the one-year default loss distribution, although the analysis can easily be extended to migration losses or to multiple steps.

Consider a portfolio of *m* obligors. To simplify, defaults are modelled using a one-factor Vasicek model (a two-state form of the CreditMetrics model); this is the same model underlying the regulatory proposal of the Basel committee (Basel Committee on Banking Supervision 2001, Wilde 2001). Thus, for a given obligor, its *obligor creditworthiness index* in one year, Y, is represented by a standard normal variable (i.e., with zero mean and unit variance). We assume that Y is driven by a single systemic factor, Z, through a linear factor model:

$$Y = \beta Z + \sqrt{1-\beta^2}\varepsilon \qquad (1)$$

where $Z \sim N(0, 1)$, β is the "sensitivity" of the index to the systemic factor, and ε is an independent, standard normal variable representing the obligor-specific, or idiosyncratic, component.

A default in the model occurs when the creditworthiness index, Y, falls below a given boundary. Since the index is a standard normal variable, we can express the unconditional probability of default of the obligor as

$$\bar{p} = Pr\{Y < \alpha\} = \Phi(\alpha), \qquad (2)$$

where Φ denotes the normal cumulative density function and α is the unconditional default threshold. If the default probability is known (e.g., from the rating of the obligor), the unconditional threshold can be calculated by taking the inverse of Equation 2, $\alpha = \Phi^{-1}(\bar{p})$. From Equations 1 and 2, the default probability of the obligor *conditional* on a given scenario of the systemic factor is

$$P(Z) = \Phi\left(\frac{\Phi^{-1}(\bar{p}) - \beta Z}{\sqrt{1-\beta^2}}\right). \qquad (3)$$

In a homogeneous portfolio, every counterparty is statistically identical, which means that they all have the same weight, β, on the systemic factor. Thus, β^2 is the systemic component of the creditworthiness variance. We further refer to β^2 as the *credit correlation* or the "asset correlation."

The losses from a given obligor, should a default occur (i.e., the product of its exposure and its LGD), can be stochastic and dependent on the state of the world. Denote these losses by *V*(X), where X is a vector of "market" state variables, or factors, which can also be correlated to the systemic credit-risk factor Z. For simplicity we have kept the losses deterministic *conditional* on the (systemic) state of the world, although it is straightforward to add an idiosyncratic component. A *scenario*, or state of the world, is thus given by the joint outcome (X, Z) of the market factors and the systemic factor. We will assume that (X, Z) follow a joint normal distribution.

The portfolio model can then be solved as follows:

- Draw a large sample of scenarios (X, Z) from their joint distribution
- Under each Scenario:
    - compute the conditional (systemic) default losses for each obligor *V*(X)
    - compute the conditional default probabilities for each obligor *P*(Z)
    - compute the *conditional portfolio loss distribution* (note that, here, a variety of computational tools are available since conditional defaults are now independent events)
- Obtain the *unconditional* portfolio loss distributions by averaging over all scenarios.

The credit capital can be obtained from a given measure on the unconditional loss distribution (e.g., CreditVaR at a 99.9% level). As shown in the appendix, for this simple, one-factor model, various measures, such as expected losses and their variance, might be computed analytically when there is independence between market and credit risk. In general, however, one must use simulation or other approximation techniques.

The *systemic risk* of the portfolio, which can be a good approximation for large, well-diversified portfolios, can be easily obtained by applying the Law of Large Numbers (LLN). Formally, the systemic losses can be defined as the actual losses of an *asymptotically fine-grained portfolio* (we obtain the portfolio by dividing every exposure into q identical exposures and taking the limit as q becomes very large). In this case, conditional on a given scenario, the loss distribution collapses to a single point (its expected loss) and all higher moments vanish. This is a direct consequence of the LLN and the property of scenario conditional independence. Thus, the systemic loss distribution is given by the distribution of expected losses over the scenarios.

Case study

The objective of this case study is to illustrate how stochastic (and correlated) exposures and LGDs affect credit capital. The contribution of stochastic exposure/LGDs depends mainly on four factors:

- the volatility/dispersion of individual exposures/LGDs
- portfolio granularity (i.e., the level of portfolio diversification)
- market correlations (codependence of exposures/LGDs)
- market and credit correlations.

We use a derivatives portfolio for this exercise, which contains market-driven exposures. However, the conclusions are fairly general for either trading or banking book portfolios with exposures or LGDs that are stochastic.

In order to highlight the main points and have a small number of controlled parameters, we choose simple, stylized homogeneous portfolios of interest-rate swaps. Also, to keep the problem of correlations simple, and to show the impact on regulatory recommendations, we use a one-factor creditworthiness model for each counterparty. A full, multi-factor interest-rate model, calibrated to historical data, is used to capture realistic credit exposures and model realistically the size of exposures' volatility.

Next, we describe the portfolio as well as the market and credit data, followed by a presentation of the details of the analysis and the results.

Instruments, credit and market data

We consider various homogeneous portfolios where all counterparties are rated BB, with an unconditional, one-year default probability of 1%. Each counterparty in the portfolio contains a single position in one of two instruments: a fixed receiver or a payer, USD interest-rate swap. We assume that the swaps were previously issued and have a remaining three-year maturity and a swap notional of one thousand USD. The coupon on the fixed leg is 6.636% (on the day of the analysis, the mark-to-market for the receiver and payer swaps are plus or minus \$30, respectively). Recoveries in the event of default are assumed to be zero ($LGD = 100\%$), and thus each swap exposure $V(X)$ represents the losses conditional on a given scenario (X,Z).

To capture stochastic swap exposures, the model assumes that interest rates follow a multi-factor, mean-reverting process as described, for example, in Reimers and Zerbs (1999). The model uses five principal components, and calibrate parameters using three years of historical data (the date of the calibration is September 1, 2000). We denote by ρ the *market-credit correlation*; more precisely ρ denotes the negative of correlation of the single credit driver, Z, and the first principal factor of the model for interest-rates evolution through time. (Positive ρ means the positive correlation between the distribution of credit exposures of the receive fixed leg of the swap and the credit driver.) In order to understand the impact of the correlation between market and credit risk, this correlation is varied in different experiments. Furthermore, in order

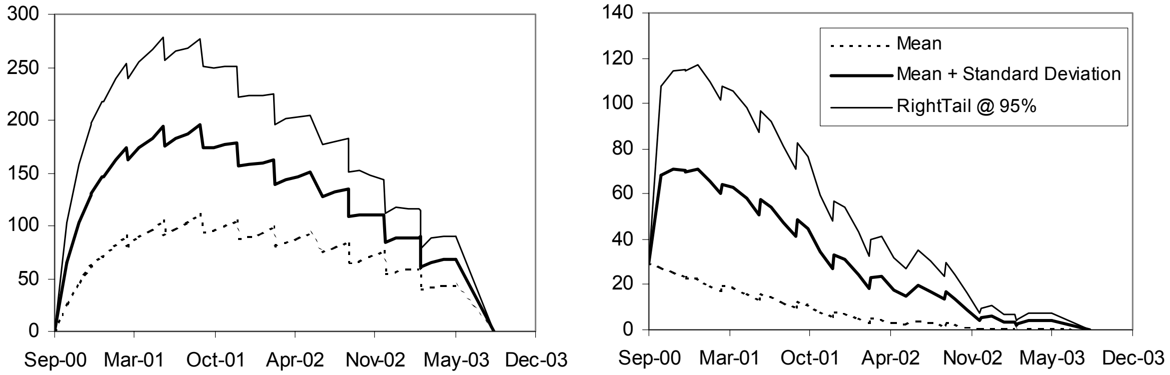

Figure 1: Swap exposures profiles ($USD)

to understand the impact of diversification and credit correlations, the weight β is also varied in different experiments.

	Pay Fixed	Receive Fixed
Expected Exposure	89	13
Standard Deviation	77	35
Quantile (95%)	228	89
Quantile (99%)	288	167
Quantile (99.9%)	368	265

Table 1: One-year swap exposures statistics ($USD)

Figure 1 shows the exposure profiles for the three-year swaps obtained from a Monte Carlo simulation, and Table 1 presents some statistics of their one-year exposure distribution. Their expected exposures at one year are about $89 and $13 for the payer and the receiver swap, respectively; their 99.9% exposures range from almost $370 to $265. Note that while the actual exposure today of the payer swap is zero (since it is out of the money) and that of the receiver is $30, the payer's exposure in one year is much higher (e.g., the mean exposure is almost seven times larger and the 99.9% exposure is almost 40% larger). In this case, this is a direct result of the mean-reversion levels in the model, which are well below the spot interest rates. Hence, the payer (receiver) swap tends to be in the money (out of the money) a substantial part of the time in the simulation.

Credit capital for finite portfolio

In order to assess the impact of stochastic exposures for a given portfolio, we compute the default loss distributions given by two portfolio models:

- **Deterministic exposure (DE)**: a model that makes the common assumption that exposures are *deterministic*. As is standard practice, the loan-equivalent exposures used as inputs to the model are given by the expected exposures in Table 1.
- **Stochastic exposure (SE)**: a portfolio model with *stochastic* exposures; these exposures are calculated through a Monte Carlo simulation.

Note that to capture the losses over one year properly, a multi-step portfolio model is required. The need for an integration through time is evident from Figure 1, which clearly shows the time dependency of the exposures. Hence, the precise timing of default during the one-year measurement horizon can have a substantial impact. In order to keep the problem simple and focus on the impact of exposure volatilities and correlations, we use a single-step model. The importance of multi-step credit models will be addressed in a separate paper.

Base case. Consider a homogeneous portfolio of 72 counterparties, each with a payer swap. Both credit and market-credit correlations are set equal to 25%. Table 2 presents some relevant statistics for the credit loss distributions with both stochastic and deterministic exposures.

	Deterministic Exposures	Stochastic Exposures
Expected Loss	64	77
Standard Deviation	137	229
Credit VaR (95%)	292	347
Credit VaR (99%)	558	1,018
Credit VaR (99.9%)	1,181	2,317
Expected Shortfall (95%)	421	852
Expected Shortfall (99%)	729	1,666
Expected Shortfall (99.9%)	1,420	3,275

Table 2: Loss statistics **L** ($USD). Base case (72 payer swaps, $\beta^2 = 0.25$, $\rho = 0.25$)

The impact of the market-credit correlation is already visible on the expected losses, where the model with stochastic exposures already shows over 20% increase. The impact on dispersion measures is much higher (e.g., the measures in the tail of the distribution more than double when exposures are considered stochastic). This difference arises from the joint impact of the exposure volatility, the market correlations (which are 100%) and market-credit correlations.

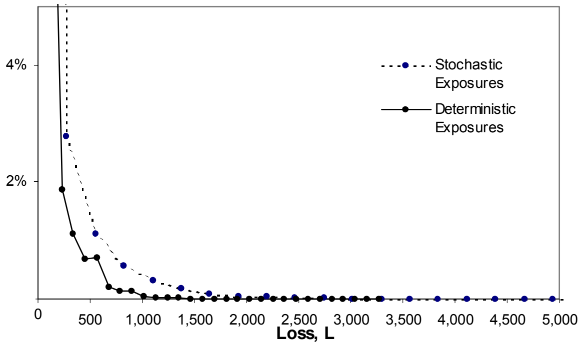

Figure 2: Tails of portfolio loss distributions; base case ($USD)

Figure 2 depicts further the tails of both loss distributions. As expected, the tail is much thicker, and extends much further, when the stochastic exposures are considered. Although not shown in the picture, it is easy to see that the tail of the deterministic case is truncated at losses of about $6,400 (the maximum possible losses when all 72 swaps default for a loss of $89 each). In contrast, with stochastic exposures, such a loss could occur when only about one-quarter of the swaps default on an extreme market move (see the 99.9% exposures in Table 1). Alternatively, the losses, if the joint event of having a 99.9% market move and all counterparties default, would produce losses over four times larger than the maximum loss with deterministic exposures.

The large differences in the statistics in Table 3 clearly show the impact of stochastic exposures. While some loss statistics are over 200% higher for this portfolio of payer swaps, the percent difference in the models is much larger for a similar portfolio of receiver swaps (with the market-credit correlation now being negative). While we do not present the result here, this outcome can be easily understood by noting that the relative difference between the mean and tail-swap exposures in Table 1 is much larger for the receiver swap (e.g., the 99.9% tail exposure is over 20 times the expected exposure for the receiver, but only about four times for the payer).

Impact of credit correlations. Next, we test the impact of credit correlations on the portfolio losses for both models in the base case. Figure 3 presents various loss statistics for different values of the credit correlation $\beta^2 = 0, 0.25, 0.50$. While the credit correlation does not affect expected losses for the deterministic case, the impact is already important for stochastic exposures. It is also evident from the figure that the impact of

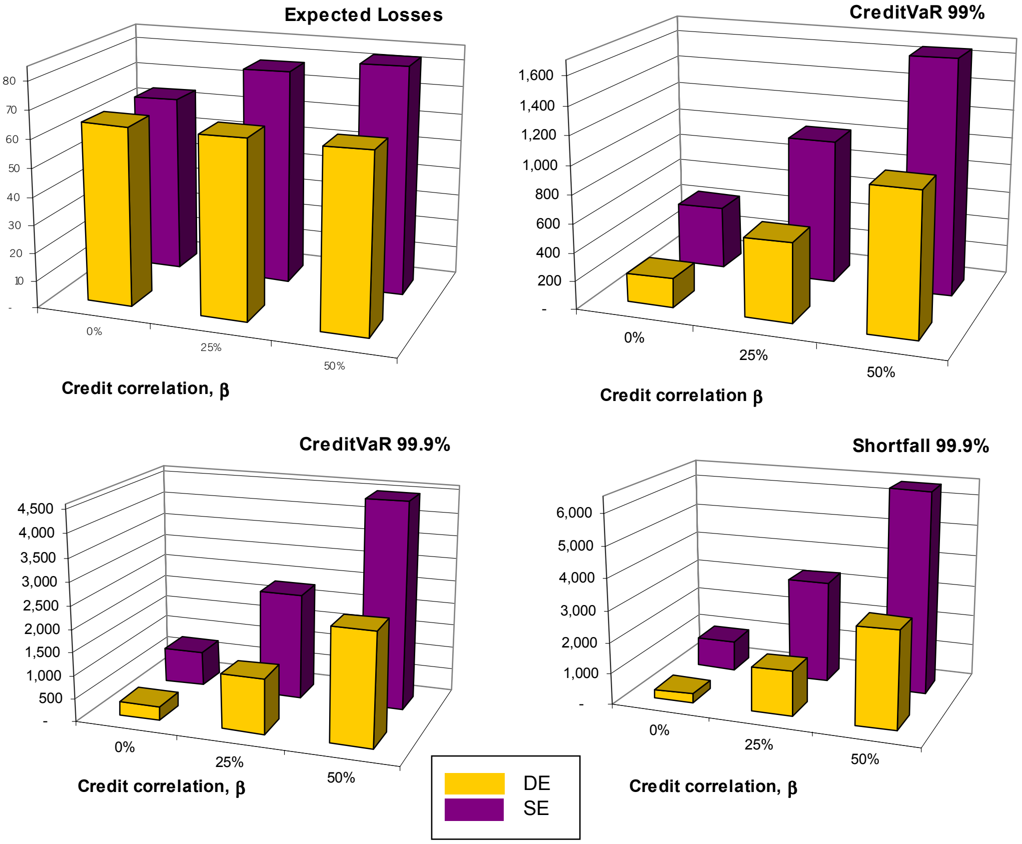

Figure 3: Impact of credit correlations on portfolio losses; base case ($\rho = 0.25$)

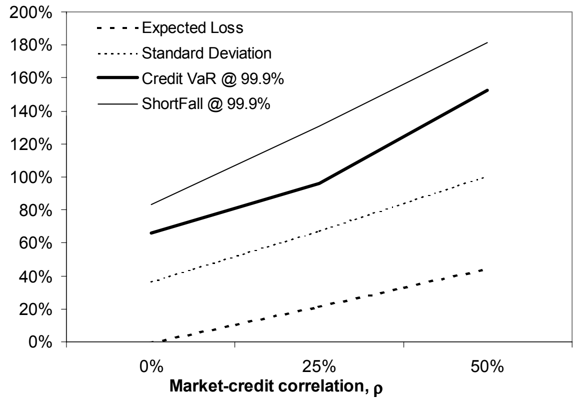

Figure 4: Impact of market-credit correlations on portfolio, Δ (%); ($\beta^2 = 0.25$)

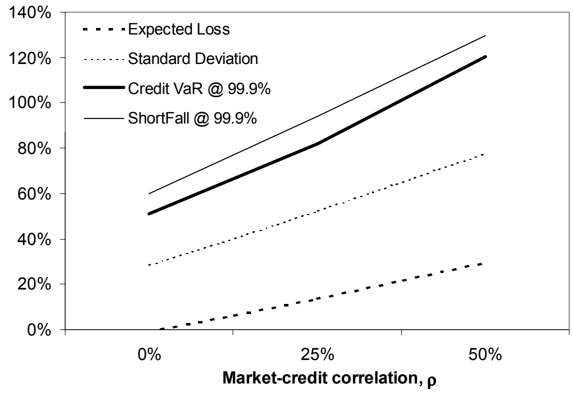

Figure 5: Impact of credit correlations on portfolio with fixed and receiver swaps Δ (%); ($\beta^2 = 0.25$)

the credit correlations is accentuated for all risk measures in the case of stochastic exposures.

Impact of market-credit correlations. Figure 4 depicts the impact on the base portfolio-credit losses of the market-credit correlation. The graph shows the percent difference between the two models for various levels of the market-credit correlation ($\rho = 0, 0.25, 0.50$). More precisely, this percent difference is given by the ratio Δ:

$$\Delta = \frac{\theta_{SE} - \theta_{DE}}{\theta_{DE}} \tag{4}$$

where *SE* denotes stochastic exposures, *DE* deterministic exposures and θ is a given statistic on the distribution (e.g., mean or CreditVaR).

When market and credit risks are uncorrelated, stochastic exposures have no impact on the expected losses. The impact on dispersion measures, however, is substantial (over 40% for CreditVaR and 80% for the expected shortfall). As expected, the difference between the models increases dramatically with correlations. For example, CreditVaR almost triples at a 50% correlation, and expected losses are already 40% higher. This outcome shows that the accurate modelling of stochastic exposures and correlations has a substantial impact on both credit reserves and economic capital.

Market-hedged portfolio. Note that, from the perspective of market risk diversification, this portfolio represents the worse case since all counterparty exposures are perfectly correlated. Thus, for example, when interest rates fall, all the exposures simultaneously rise. Consider now a second "market hedged" portfolio with 72 counterparties, half of which contain a single payer swap, and the second half a single receiver swap. We also assume that all counterparties have identical creditworthiness models (1% default probability and the same systemic component).

This is a portfolio that is hedged for market movements and does not contain, as a whole, market risk. Thus, whether interest rates rise or fall, half of the counterparties have positive exposures, while the other half shows no credit exposure. It is not possible, however, in this case, to have simultaneous defaults of more than one-half of the counterparties, regardless of the state of the world. Figure 5 compares the loss statistics for a portfolio with stochastic exposures, and one where exposures are assumed to be deterministic for various market-credit correlations. Once again, the credit correlation is assumed to be equal to 25% as in the base case ($\beta^2 = 0.25$).

While the modelling assumption on the exposures has a smaller impact in this case, it is still substantial and the graph patterns are the same. However, it is not easy to predict the results in this case because there are several effects moving losses in different directions. For example, the "market hedges" tend to diminish the loss probabilities in any given scenario since only one-half of the counterparties would default on their swap even if their creditworthiness were in distress. Alternatively, losses may be incurred in almost all scenarios since one of the two swaps will be in the money. Also, adding the receiver swaps into the portfolio increases the model differences in this case since, as was explained earlier, the percent difference between the expected and tail exposures is much larger for these instruments.

Impact of portfolio size on credit-risk capital

So far, we have shown the impact of stochastic exposures for "small" portfolios. However, it is important to understand the impact of this modelling assumption as the portfolio size increases.

The regulatory Basel proposal is based on decomposing risk into a systemic component (arising from the assumption that the portfolio is infinitely granular) and a granularity adjustment to correct for finite portfolios that are not perfectly diversified. As described in the theoretical papers underlying the regulation (e.g., Gordy 2001), the computation of systemic losses requires only expected exposures and LGDs when the true exposures and LGDs are deterministic or stochastic, but are independent of each other (and of the credit events). This is basically also a direct consequence of the LLN. Thus, the ISDA's response to the proposal (ISDA 2001) suggests allowing for internal models to measure the *loan equivalent exposure* (LEE)

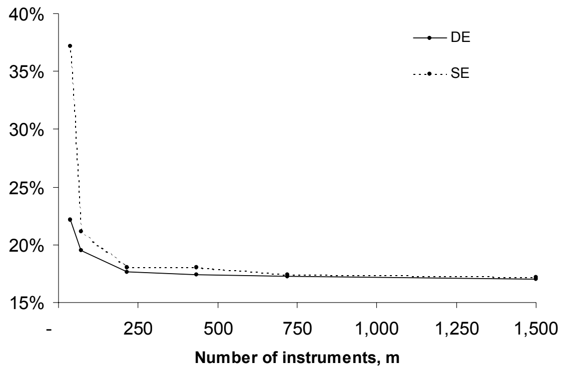

Figure 6: Credit losses @ 99.9% for a portfolio with uncorrelated exposures (% of total expected portfolio exposures)

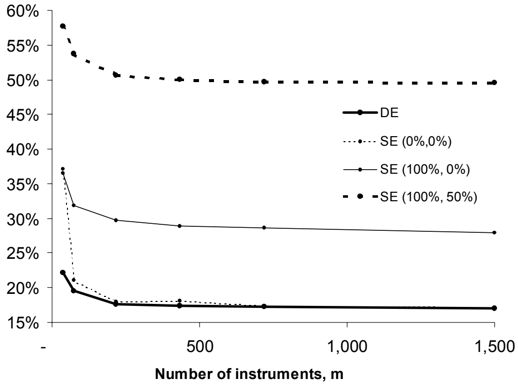

Figure 7: Credit losses @ 99.9% for a portfolio with uncorrelated exposures (% of total expected portfolio exposures)

for derivatives portfolios and using the expected counterparty portfolio exposure as the LEE measure.

It is thus important to understand the impact of two types of factors on the capital calculations for a finite portfolio with true stochastic exposures or LGDs. They are: the level of granularity, and correlations between exposures/LGDs and between market and credit events. First, it is important to recognize the level of *granularity*, and, hence, the fact that exposure volatilities do not diversify away and must be accounted for. This analysis is performed in detail in Gordy (2001) for uncorrelated exposures/LGDs. He shows that, for a homogeneous portfolio, the capital asymptotically converges as a function of the inverse of the number of obligors. This analysis forms the basis of the proposal for the granularity adjustment in the regulatory paper.

The second set of factors comprises the *correlations* between exposures/LGDs and between market and credit events. In practice, exposures/LGDs can be heavily correlated, for example, in derivatives portfolios, portfolios with collateral or when other systemic economic factors come into play. There are also many cases in practice where market-credit correlations have been substantial. Generally referred to as *wrong-way exposures,* such cases have been an important cause of substantial losses.

Consider now the base portfolio, but assume that the counterparty exposures are independently drawn from the distribution of the payer swap from Table 1 (one can think of this as having each swap in a different, uncorrelated currency, for example). As in the base case, the credit correlation is $\beta^2 = 0.25$ (and, of course, in this case $\rho = 0$ for all swaps). Figure 6 shows 99.9% credit losses as a function of the number of obligors in the portfolio. To simplify the comparison, losses are expressed as a percent of the total expected portfolio exposure (for a homogeneous portfolio, this is given by the product of the number of obligors and the expected swap exposure in one year).

Figure 6 shows the quick convergence of the credit losses with SE to those of the DE. In the limit, the systemic losses are about 17% of the total expected exposure. When exposures are stochastic, the total losses are more than double the systemic losses for a portfolio with 36 counterparties, and are almost 25% higher for the base portfolio (72 counterparties). In contrast, when exposures are deterministic these numbers fall to 30% and 15%, respectively. The convergence of both cases to the systemic losses illustrates how the diversification, coming from increasing the size of a portfolio, eventually cancels out the effect of individual exposure volatility of each counterparty.

The same convergence to systemic losses is not observed when the exposures are not independent. Figure 7 shows the 99.9% credit losses as a function of the number of obligors for two additional cases that present exposure correlations, and contrasts the results without correlations as presented above. In the first case denoted SE(100%, 0%), we assume that the exposures are perfectly correlated as in the base case, but the market and credit correlations are set to zero (independent market and credit), and the second case further assumes a market-credit correlation of 50% (SE(100%, 50%).

Correlations clearly have a great impact for both finite portfolios and systemic losses. In particular, the exposure correlations (on their own) increase the systemic losses by almost 65%, while the joint effect of exposure correlations and market-credit correlations almost triple these losses. These results have clear implications for regulatory internal models. In particular, short of allowing full portfolio credit models, which properly account for correlations, then the systemic losses as presented in the regulatory model require, in addition, an adjustment for two effects: exposures/LGD correlations (i.e., market correlations) and market-credit correlations. The granularity adjustment would then be added on top of the total systemic loss, and should correct for exposure volatilities and correlations. As shown in this paper, these effects are captured in a straightforward way with an integrated market and credit-simulation model.

Concluding remarks

This paper presents a stylized case study that demonstrates the impact of stochastic and correlated exposures on credit capital. In this sense, four factors have a substantial effect on credit losses: exposure (market) volatility, credit correlations, market-credit correlations, and portfolio granularity. The results have important implications for both economic and regulatory capital, and their implications apply as well when LGDs are stochastic.

In terms of regulation, stochastic exposures and their correlations cannot be ignored even for infinite granular portfolios and systemic losses. Short of allowing full internal portfolio credit-risk models, there are two alternatives to correct for this in the current regulatory proposal. The first is to add a “systemic correlation adjustment” at the portfolio level that is similar to the granularity adjustment. In this case, internal models for computing loan-equivalent exposures can be based on expected exposure measures, as has been suggested.

In the second case, instead of adjusting the overall portfolio losses, loan equivalent exposures may be computed directly from an internal model that corrects for correlations. Analytical solutions might be possible for some simple cases only; otherwise, a simulation model, comparable to the one presented herein, can be used. Analytical approximations for this type of solution are outside the scope of this paper.

Although the examples developed in this paper use derivatives portfolios, the methodology and implications are quite general. Thus, there are substantial benefits in using a flexible, integrated market and credit-simulation model for portfolios with stochastic exposures and LGDs, spreads and portfolio collateral, and other mitigation techniques.

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Appendix: Some analytical solutions

Notation

Consider a homogeneous portfolio, and denote by V, the stochastic counterparty exposures at the measurement horizon. Further denote by $\bar{V}$ and σ_V^2, respectively, the mean and variance of the distribution of stochastic exposures.

The credit driver $Z \sim N(0, 1)$ is transformed into the conditional default probabilities P as in Equation 3. N is the number of scenarios (the length of the vector of joint drawings from Z and the market factors driving exposures).

In a one-factor creditworthiness model, the variance of the conditional default probabilities, σ_P^2, is given by:

$$\sigma_P^2(\beta, \bar{p}) = \int_{-\infty}^{\infty} P(Z)^2 d\Phi(Z) - (\bar{p})^2 . \qquad \text{(A1)}$$

We further denote the parameters of the distribution of perfectly correlated counterparty exposures by the superscript $^{||}$, and those of the portfolio of independent exposures by $^{\perp}$.

Characteristic function

Let ν represent the number of defaults for a portfolio of m counterparties, in the case of the homogenous portfolio, ν is binomially distributed.

Denote the characteristic function (Ch.f) of portfolio losses, L, by $\varphi_L(s) = Ee^{isL}$.

Conditional on the value of the credit driver, the portfolio of *perfectly correlated counterparty exposures* has the following CF of losses:

$$\varphi^{\|}_L(s) = Ee^{isL} = Ee^{is\nu V} = \sum_{k=0}^{m} \pi_k(Z)\varphi_V(ks), \tag{A2}$$

where π_k is the probability

$$\pi_k = \pi(\nu = k) = \binom{m}{k} P(Z)^k (1 - P(Z))^{m-k}.$$

Similarly, the Ch.f for a portfolio of *independent counterparty exposures* is given by:

$$\varphi^{\perp}_L(s) = Ee^{isL} = Ee^{is\sum_{j=1}^{\nu} V_j} = \sum_{k=0}^{m} \pi_k(Z)\varphi_V(s)^k. \tag{A3}$$

Expected credit losses

Using the basic properties of Ch.fs, we can compute the moments of the loss distribution. The expected loss of the portfolio of *perfectly correlated counterparties* is then equal to:

$$\begin{aligned} E[L^{\|}] &= -i\frac{d\varphi^{\|}_L(s)}{ds}\bigg|_{s=0} \\ &= \sum_{k=0}^{m} \pi_k(Z)k\varphi_V'(ks) \\ &= E[\nu E[V]] \\ &= mP(Z)V(X) \end{aligned} \tag{A4}$$

and the expected loss of the portfolio of *independent counterparties:*

$$\begin{aligned} E[L^{\perp}] &= -i\frac{d\varphi^{\perp}_L(s)}{ds}\bigg|_{s=0} \\ &= \sum_{k=0}^{m} \pi_k(Z)k\varphi_V(s)^{k-1}\varphi'_V(s) \\ &= E[\nu E[V]] \\ &= mP(Z)V(X). \end{aligned} \tag{A5}$$

Then, for both cases, the mean portfolio losses for the sample of the length N are equal to

$$E[L_{SE}] = \frac{m\sum_j P_j V_j}{N}, \tag{A6}$$

where SE stands for stochastic exposures. If exposures are assumed to be deterministic, and are given by the mean exposure over the market scenarios, the mean losses are then simply given by:

$$E[L_{DE}] = \bar{V}m\bar{p}. \tag{A7}$$

Variance of credit losses

For perfectly correlated exposures, the second moment of portfolio losses is:

$$\begin{aligned} E[L^{\|2}] &= -\frac{d^2}{ds^2}\varphi^{\|}_L(s)\bigg|_{s=0} \\ &= \sum_{k=0}^{m} \pi_k(Z)k^2\varphi''_V(ks) \\ &= E[\nu^2 E[V^2]] \\ &= (m(m-1)P(Z)^2 + mP(Z))E[V^2]. \end{aligned} \tag{A8}$$

The variance of losses is given by

$$\begin{aligned} \sigma^2 L^{m\,\|}_{SE} &= \frac{1}{N}\left[m\sum_j P_j V_j^2((m-1)P_j + 1) - m\sum_j P_j V_j\right] \\ &= m^2(E[(VP)^2] - E[VP]^2) \\ &\quad + m(E[V^2P] - E[(VP)^2]). \end{aligned} \tag{A9}$$

If market and credit events are independent, this expression can be further simplified as:

$$\begin{aligned}\sigma^2 L_{SE}^{m\,\parallel}{}_{ind} &= m^2(E[V^2]E[P^2]) \\ &\quad + mE[V^2](E[P]-E[P^2]) \\ &= m^2(\bar{V}^2\sigma_{\bar{p}}^2 + \sigma_V^2\bar{p}^2 + \sigma_V^2\sigma_{\bar{p}}^2) \\ &\quad + m(\bar{V}^2 + \sigma_V^2)(\bar{p} - \bar{p}^2 - \sigma_P^2).\end{aligned} \tag{A10}$$

This expression is simplified for the case where the exposures are deterministic:

$$\begin{aligned}\sigma^2 L^m{}_{DE} &= \frac{\bar{V}^2 m \sum_j P_j(1 - P_j + mP_j)}{N} - (\bar{V}m\bar{p})^2 \\ &= \sigma_P^2\bar{V}^2 m^2 + \bar{V}^2 m(\bar{p} - \bar{p}^2 - \sigma_P^2).\end{aligned} \tag{A11}$$

For the case of independent exposures, the second moment of portfolio losses is:

$$\begin{aligned}E[L^{\perp 2}] &= -\frac{d^2}{ds^2}\varphi_L^{\perp}(s)\bigg|_{s=0} \\ &= \sum_{k=0}^{m} \pi_k(Z)k(k-1)\varphi_V(s)^{k-2}\varphi_V'(s) \\ &\quad + \sum_{k=0}^{m} \pi_k(Z)k\varphi_V(s)^{k-1}\varphi_V''(s) \\ &= E[\nu^2](E[V])^2 \\ &\quad -E[\nu](E[V])^2 + E[\nu]E[V^2] \\ &= (E[V])^2(m(m-1)P(Z)^2) + mP(Z)E[V^2].\end{aligned} \tag{A12}$$

The variance of credit losses for independent market and credit events are then given by:

$$\begin{aligned}\sigma^2 L_{SE\,ind}^{m\perp} &= \bar{V}^2\left(\frac{\sum_j P^2{}_j(m(m-1))}{N}\right) \\ &\quad + m\frac{\sum_j P_j(\bar{V}^2 + \sigma_V^2)}{N} - m^2\bar{V}^2\bar{p}^2 \\ &= m^2\bar{V}^2\sigma_{\bar{p}}^2 + m\bar{p}(\bar{V}^2 + \sigma_V^2) \\ &\quad -m\bar{V}^2(\bar{p}^2 + \sigma_{\bar{p}}^2).\end{aligned} \tag{A13}$$

Table A1 summarizes the expressions for the variance of losses for different assumptions of correlations between counterparty exposures and market and credit risk.

	Total Loss Distribution	**Systemic Loss Distribution** $(m \to \infty)$
Deterministic Exposures	$\sigma^2 L^m_{DE} = \sigma^2 L^{\infty}{}_{DE} + \frac{\bar{V}^2(\bar{p}-\bar{p}^2-\sigma_P^2)}{m}$	$\sigma^2 L^{\infty}_{DE} = \sigma_P^2 \bar{V}^2$
Stochastic Exposures	$\sigma^2 L^m_{SE} = \sigma^2 L^{\infty}{}_{SE} + \frac{E[V^2 P] - E[(VP)^2]}{m}$	$\sigma^2 L^{\infty}_{SE} = E[(VP)^2] - E[VP]^2$
Independent Market and Credit		
Stochastic Exposures (Counterparties are perfectly correlated)	$\sigma^2 L^{m\ \Vert}_{SE\ \ ind} = \sigma^2 L^{\infty\ \Vert}_{SE\ \ ind} + \frac{(\bar{V}^2 + \sigma_V^2)(\bar{p}-\bar{p}^2-\sigma_P^2)}{m}$	$\sigma^2 L^{\infty\ \Vert}_{SE\ \ ind} = \sigma^2 L^{\infty}_{DE} + \sigma_P^2\sigma_V^2 + \bar{p}^2\sigma_V^2$
Stochastic Exposures (Counterparties are independent)	$\sigma^2 L^{m\perp}_{SE\ ind} = \sigma^2 L^{\infty}_{DE} + \frac{\bar{p}\sigma_V^2 + \bar{V}^2(\bar{p}-\bar{p}^2-\sigma_P^2)}{m}$	$\sigma^2 L^{\infty\perp}_{SE\ ind} = \sigma^2 L^{\infty}_{DE}$

Table A1: Variance of loss distribution (per instrument)